Quantum state merging with bound entanglement

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Quantum state merging is one of the most important protocols in quantum information theory. In this task two parties aim to merge their parts of a pure tripartite state by making use of additional singlets while preserving coherence with a third party. We study a variation of this scenario where the merging parties have free access to PPT entangled states, and the total quantum state shared by all three parties is not necessarily pure. We provide general conditions for a state to admit perfect merging, and present a family of fully separable states which cannot be perfectly merged if the merging parties have no access to additional singlets. We also show that for pure states the conditional entropy plays the same role as in standard quantum state merging, quantifying the amount of quantum communication needed to perfectly merge the state. While the question whether the protocol considered here exhibits the strong converse property is left open, it is shown that for a significant amount of quantum states the merging fidelity vanishes asymptotically.

PACS numbers: 03.67.Ac, 03.67.Hk, 03.67.Lx, 03.67.Mn

Introduction. Quantum state merging can be understood as a game involving three players, which we will call Alice, Bob and Charlie in the following. Initially, they share a large number of copies of a joint pure state $|\psi\rangle = |\psi\rangle^{ABC}$, and the aim of Bob and Charlie is to merge their parts of the state on Charlie's side while preserving the coherence with Alice. For achieving this, Bob and Charlie have access to additional singlets. Taking into account that singlets are considered as an expensive resource in quantum information theory, the main question of quantum state merging can be formulated as follows: How many singlets are required for perfect asymptotic merging per copy of the state $|\psi\rangle$? The answer to this question was found in [1, 2]: the minimal number of singlets per copy is given by the conditional entropy $S(\rho^{BC}) - S(\rho^C)$.

Noting that the conditional entropy can be positive or negative, it is surprising that it admits an operational interpretation in both cases. In particular, if the conditional entropy is positive, Bob and Charlie will require $S(\rho^{BC}) - S(\rho^C)$ singlets per copy for perfectly merging the total state $|\psi\rangle$ in the asymptotic limit, and perfect merging cannot be accomplished if less singlets are available. On the other hand, if $S(\rho^{BC}) - S(\rho^C)$ is negative, Bob and Charlie can asymptotically merge the state $|\psi\rangle$ without any additional singlets by only using local operations and classical communication (LOCC). Moreover, Bob and Charlie can gain additional singlets at rate $S(\rho^C) - S(\rho^{BC})$, and store them for future use [1, 2].

Another important concept in quantum information theory is the framework of entanglement distillation [3–5]. One of the most surprising features in this context is the phenomenon of bound entanglement: there exist entangled states from which no singlets can be distilled [6]. Moreover, it is known that all states with positive partial transpose (PPT) are nondistillable [6], while it is still an open question if there exist bound entangled states with nonpositive partial transpose (NPT) [7].

In this paper we introduce and study the task of *PPT quantum state merging (PQSM)*. Similar to standard quantum state merging, PQSM can be considered as a game between three players who share a joint mixed state $\rho = \rho^{ABC}$. The aim of the game for Bob and Charlie is to merge their parts of the state ρ on Charlie's side while preserving coherence with Alice. In contrast to standard quantum state merging, Bob and Charlie can use unlimited amount of PPT entangled states, see Fig. 1 for illustration. The situation where Bob and Charlie do not have access to PPT entangled states is known as LOCC quantum state merging (LQSM), and has been introduced recently in [8].

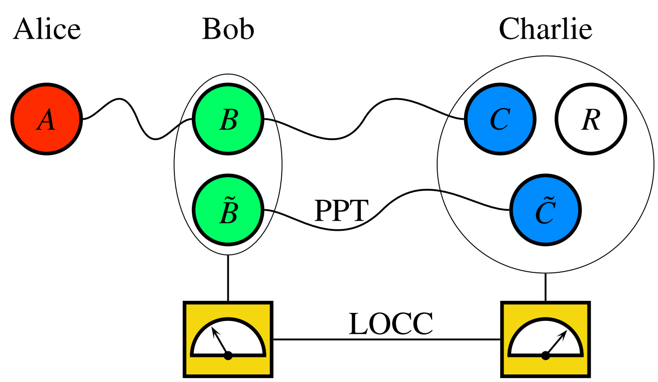

Figure 1. PPT quantum state merging (PQSM). Alice, Bob and Charlie initially share a joint state $\rho = \rho^{ABC}$. Bob and Charlie aim to merge Bob's part of ρ on Charlie's side, while preserving coherence with Alice. For this, Bob and Charlie have access to arbitrary PPT states $\mu_{\mathrm{PPT}} = \mu_{\mathrm{PPT}}^{\tilde{B}\tilde{C}}$, and can perform local operations on their parts and communicate the outcomes via a classical channel. The register R in Charlie's hands serves as storage: in the ideal case, the final state σ_f^{ACR} is equivalent to ρ^{ABC} up to relabeling B and R.

Before we discuss the concept of PPT quantum state merging and present our main results, we will introduce PPT assisted LOCC operations in the following.

PPT assisted LOCC. For a tripartite state ρ^{ABC} shared between Alice, Bob and Charlie, a PPT assisted LOCC protocol performed by Charlie and Bob will be denoted by Λ_{PPT} and has

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the following form:

$$\Lambda_{\mathrm{PPT}}\left(\rho^{ABC}\right) = \mathrm{Tr}_{\tilde{B}\tilde{C}}\left[\Lambda_{\mathrm{LOCC}}\left[\rho^{ABC}\otimes\mu_{\mathrm{PPT}}^{\tilde{B}\tilde{C}}\right]\right]. \tag{1}$$

Here, $\mu_{\mathrm{PPT}}^{\tilde{B}\tilde{C}}$ is an arbitrary PPT state shared by Bob and Charlie, and Λ_{LOCC} is an LOCC protocol between them, see Fig. 1.

We also introduce PPT distillable entanglement D_{PPT} as the number of singlets which can be asymptotically obtained from a state via PPT assisted LOCC. This quantity is in full analogy to the standard distillable entanglement [3] that quantifies the number of singlets which can be distilled via LOCC only. We will denote the latter by D_{LOCC}.

If we further introduce the PPT and LOCC entanglement cost C_{PPT} and C_{LOCC} as the entanglement cost for creating a state via the corresponding set of operations, we immediately obtain the following inequality:

$$D_{\mathrm{LOCC}}(\rho) \leq D_{\mathrm{PPT}}(\rho) \leq C_{\mathrm{PPT}}(\rho) \leq C_{\mathrm{LOCC}}(\rho). \tag{2}$$

This relation follows by noting that PPT assisted LOCC is more general than LOCC only, and by the fact that the PPT entanglement cost cannot be below the PPT distillable entanglement. Moreover, the above inequality immediately implies that for pure states all these quantities coincide, and are equal to the von Neumann entropy of the reduced state. In the following, we will also use the logarithmic negativity [9, 10]

$$E_n(\rho) = \log_2\|\rho^{T_A}\|, \tag{3}$$

where T_A denotes partial transposition, and $\|M\| = \mathrm{Tr}\sqrt{M^\dagger M}$ is the trace norm of M. The logarithmic negativity is an upper bound on D_{LOCC} [10].

We further note that PPT assisted LOCC is a subclass of general PPT preserving operations. It is however not clear whether or not these two classes coincide.

PPT quantum state merging. We are now in position to introduce the aforementioned task of PPT quantum state merging (PQSM). In this task, Bob and Charlie aim to merge their parts of the total state $\rho = \rho^{ABC}$ by using PPT assisted LOCC operations, see Fig. 1 for illustration. A natural figure of merit for this process is the fidelity of PQSM:

$$\mathcal{F}_{\mathrm{PPT}}(\rho) = \sup_{\Lambda_{\mathrm{PPT}}} F(\sigma_f, \sigma_t) \tag{4}$$

with fidelity $F(\rho,\sigma) = \mathrm{Tr}(\sqrt{\rho}\sigma\sqrt{\rho})^{1/2}$. In the above expression, the target state $\sigma_t = \sigma_t^{ACR}$ is the same as $\rho = \rho^{ABC}$ up to relabeling of the systems B and R, where R is an additional register in Charlie's hands. The final state $\sigma_f = \sigma_f^{ACR}$ shared by Alice and Charlie is given by

$$\sigma_f = \mathrm{Tr}_B\left[\Lambda_{\mathrm{PPT}}\left[\rho^{ABC}\otimes\rho^R\right]\right], \tag{5}$$

where ρ^R is an arbitrary initial state of Charlie's register R. The supremum in Eq. (4) is taken over all PPT assisted LOCC operations Λ_{PPT} between Bob's system B and Charlie's system CR, see also Fig. 1 for details. A state ρ admits *perfect single-shot PQSM* if and only if the corresponding fidelity is equal to one: $\mathcal{F}_{\mathrm{PPT}}(\rho) = 1$, and $\mathcal{F}_{\mathrm{PPT}}(\rho) < 1$ otherwise.

In the asymptotic scenario where a large number of copies of the state ρ is available, the figure of merit is the asymptotic fidelity of PQSM:

$$\mathcal{F}_{\mathrm{PPT}}^{\infty}(\rho) = \lim_{n\to\infty}\mathcal{F}_{\mathrm{PPT}}(\rho^{\otimes n}). \tag{6}$$

This quantity can be regarded as a natural quantifier for asymptotic PQSM, since a state ρ admits *perfect asymptotic PQSM* if and only if $\mathcal{F}_{\mathrm{PPT}}^{\infty}(\rho) = 1$.

Perfect asymptotic PQSM. In the following we will focus on those states $\rho = \rho^{ABC}$ which admit perfect asymptotic PQSM:

$$\mathcal{F}_{\mathrm{PPT}}^{\infty}(\rho) = 1. \tag{7}$$

In particular, perfect asymptotic PQSM is always possible if the state ρ has nonpositive conditional entropy:

$$S(\rho^{BC}) - S(\rho^{C}) \leq 0. \tag{8}$$

This follows from the fact that in this situation Bob and Charlie can achieve perfect asymptotic merging for the purification of ρ just by using local operations and classical communication [1, 2, 8]. Moreover, Eq. (8) also implies that states satisfying Eq. (7) have nonzero measure in the set of all states, since this is evidently true for states satisfying Eq. (8).

At this point, we also note that perfect asymptotic PQSM is only possible if the state ρ satisfies the following condition:

$$D_{\mathrm{PPT}}^{A:BC}(\rho) \leq D_{\mathrm{PPT}}^{AB:C}(\rho). \tag{9}$$

This follows directly from the fact that PPT distillable entanglement cannot grow under PPT assisted LOCC operations.

For states which satisfy Eq. (9) but violate Eq. (8) no conclusive statement can be made in general. One important subclass of such states are fully separable states, and it is easy to provide examples for such states which violate Eq. (8), but still can be merged via LOCC even on the single-copy level. In the following we will show that the investigation of such states can be simplified significantly. This will also lead us to a new class of fully separable states which cannot be merged via asymptotic PQSM.

Single-shot versus asymptotic PQSM. In the following, we consider the situation where the total state $\rho = \rho^{ABC}$ has positive partial transpose with respect to the bipartition $AB{:}C$. The set of these states includes the aforementioned set of fully separable states. The following theorem shows that for all such states the single-copy fidelity is never smaller than for any number of copies.

Theorem 1. *Given a tripartite state $\rho = \rho^{ABC}$ which is PPT with respect to AB:C, the following inequality holds for any $n \geq 1$: $\mathcal{F}_{\mathrm{PPT}}(\rho) \geq \mathcal{F}_{\mathrm{PPT}}(\rho^{\otimes n})$.*

This also implies that in this case the single-shot fidelity cannot be smaller than the asymptotic fidelity: $\mathcal{F}_{\mathrm{PPT}}(\rho) \geq \mathcal{F}_{\mathrm{PPT}}^{\infty}(\rho)$. We refer to the Supplemental Material for the proof.

Crucially, this result also means that perfect single-shot PQSM is fully equivalent to perfect asymptotic PQSM for all such states:

$$\mathcal{F}_{\mathrm{PPT}}(\rho) = 1 \Leftrightarrow \mathcal{F}_{\mathrm{PPT}}^{\infty}(\rho) = 1. \tag{10}$$

The importance of this result lies in the fact that it remarkably simplifies the analysis, if one is interested in the question whether a state ρ admits perfect asymptotic PQSM or not. For all such states we only need to study the single-copy situation: if perfect PQSM is not possible in the single-copy case, it is also not possible asymptotically.

As an application of Theorem 1, we will now present a general family of fully separable states which does not admit perfect asymptotic PQSM. These states are given by

$$\rho_{\rm sep}^{ABC} = \sum_{i=0}^{14} p_i \left|i\right\rangle\left\langle i\right|^A \otimes \sigma_i^{BC}, \tag{11}$$

where all probabilities p_i are nonzero, and the two-qubit states σ_i^{BC} are all separable and chosen such that their generalized Bloch vectors are all linearly independent. For the proof that such states exist and that they indeed do not allow for perfect asymptotic PQSM we refer to the Supplemental Material.

States with vanishing asymptotic fidelity. Taking into account the results discussed so far, it is natural to ask whether PQSM exhibits the *strong converse* property, which can be found in several tasks in quantum communication [11, 12], and also in standard quantum state merging [12, 13]. In particular, strong converse for PQSM would mean that the asymptotic fidelity $\mathcal{F}_{\rm PPT}^\infty$ can attain only one of two values, namely 0 or 1.

We can neither prove nor disprove this at the moment. Nevertheless, we will provide strong evidence for this in the following, showing that a significant amount of quantum states has vanishing asymptotic fidelity:

$$\mathcal{F}_{\rm PPT}^\infty(\rho) = 0. \tag{12}$$

This happens for all states which are distillable between A and BC, and at the same time have positive partial transpose in the bipartition $AB{:}C$. These two conditions are summarized in the following inequality:

$$D_{\rm LOCC}^{A:BC}(\rho) > E_n^{AB:C}(\rho) = 0. \tag{13}$$

The proof of this statement can be found in the Supplemental Material.

At this point it is also interesting to note that the asymptotic fidelity $\mathcal{F}_{\rm PPT}^\infty$ is not a continuous function of the state. This discontinuity is present even for pure states, and can be demonstrated on the following example:

$$\rho = \left|\psi\right\rangle\left\langle\psi\right|^{AB} \otimes \left|0\right\rangle\left\langle 0\right|^C. \tag{14}$$

Note that this state admits perfect PQSM whenever $|\psi\rangle$ is a product state, i.e., $|\psi\rangle = |\alpha\rangle^A \otimes |\beta\rangle^B$. In this case, perfect merging can be accomplished without any communication if Charlie prepares his register R in the state $|\beta\rangle^R$. Note however that the asymptotic fidelity $\mathcal{F}_{\rm PPT}^\infty$ vanishes for any entangled state $|\psi\rangle$, as follows directly from the above discussion.

As is further shown in the Supplemental Material, the set of states having vanishing asymptotic fidelity has nonzero measure in the set of all states. Combining these results with our previous findings, namely that states satisfying $\mathcal{F}_{\rm PPT}^\infty(\rho) = 1$ also have nonzero measure in the set of all states, this means that both of these sets have finite size. We hope that this result can serve as a starting point to prove strong converse for PQSM in general.

Absence of bound entanglement. The results presented in this work can also be applied to the scenario where Bob and Charlie do not have access to PPT entangled states. This task is known as LOCC quantum state merging (LQSM), and has been presented in [8]. The figure of merit in this case will be denoted by $\mathcal{F}_{\rm LOCC}$.

Note that the quantities $\mathcal{F}_{\rm LOCC}$ and $\mathcal{F}_{\rm PPT}$ obey the following relation:

$$\mathcal{F}_{\rm PPT}(\rho) \geq \mathcal{F}_{\rm LOCC}(\rho) \geq 2^{\frac{1}{2}\left[\mathcal{I}(\rho) - I^{A:BC}(\rho)\right]}. \tag{15}$$

Here, $I^{A:BC}$ is the mutual information between A and BC, and $\mathcal{I}$ is the concentrated information introduced in [8]. The concentrated information quantifies the maximal amount of mutual information between Alice and Charlie obtainable via LOCC operations performed by Charlie and Bob, and can be considered as a figure of merit for LQSM on its own right. The first inequality in (15) follows from the fact that PPT assisted LOCC operations are more general than LOCC operations alone. The second inequality in (15) crucially relies on recent results from [14–16], and the proof can be found in [8].

The second inequality in (15) further implies that $\mathcal{F}_{\rm LOCC}$ and $\mathcal{F}_{\rm PPT}$ are nonzero for any finite-dimensional state ρ. This follows directly by noting that the concentrated information $\mathcal{I}$ is nonnegative, and that the mutual information $I^{A:BC}$ is finite. The first inequality in (15) implies that all states with vanishing asymptotic PQSM fidelity also have zero asymptotic LQSM fidelity: $\mathcal{F}_{\rm PPT}^\infty(\rho) = 0$ implies $\mathcal{F}_{\rm LOCC}^\infty(\rho) = 0$. This means that all states ρ which fulfill Eq. (13) also have vanishing asymptotic LQSM fidelity: $\mathcal{F}_{\rm LOCC}^\infty(\rho) = 0$.

This result can be slightly generalized by using the same arguments as in the proof of Eq. (12). In particular, all states ρ which are distillable between A and BC but nondistillable with respect to $AB{:}C$ have vanishing asymptotic fidelity for LQSM, i.e.,

$$D_{\rm LOCC}^{A:BC}(\rho) > D_{\rm LOCC}^{AB:C}(\rho) = 0 \tag{16}$$

implies $\mathcal{F}_{\rm LOCC}^\infty(\rho) = 0$. For proving this, we can use the same proof as for Eq. (12), by noting that the final state shared by Alice and Charlie will never be distillable if the initial state ρ satisfies Eq. (16), and if Bob and Charlie use LOCC operations only.

At this point we also note that Eq. (16) does not guarantee vanishing PQSM fidelity. In particular, if there exist NPT bound entangled states – and it is strongly believed that this is indeed the case [7] – Bob and Charlie could use PPT entangled states to perfectly merge a state of the form

$$\rho = \left|\phi^+\right\rangle\left\langle\phi^+\right|^{AB_1} \otimes \rho_{\rm NPT}^{B_2C}, \tag{17}$$

where the particles B_1 and B_2 are in Bob's hands, $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is a maximally entangled two-qubit state, and $\rho_{\rm NPT}$ is an NPT bound entangled state with the property

that $D_{\rm LOCC}(\rho_{\rm NPT} \otimes \mu_{\rm PPT}) > 1$ for some PPT entangled state $\mu_{\rm PPT}$. Note that states $\rho_{\rm NPT}$ and $\mu_{\rm PPT}$ with the aforementioned properties exist if there are NPT bound entangled states [17]. Bob and Charlie can then use the state $\mu_{\rm PPT}$ to distill the state $\rho_{\rm NPT} \otimes \mu_{\rm PPT}$, and by applying Schumacher compression [18] to achieve $\mathcal{F}^{\infty}_{\rm PPT}(\rho) = 1$.

We also note that all states ρ which fulfill the condition (8) admit perfect asymptotic LQSM [8], which also implies that states with $\mathcal{F}^{\infty}_{\rm LOCC}(\rho) = 1$ have nonzero measure in the set of all states. Moreover, a state ρ admits perfect asymptotic LQSM only if it satisfies the following condition:

$$D^{A:BC}_{\rm LOCC}(\rho) \leq D^{AB:C}_{\rm LOCC}(\rho). \tag{18}$$

Similar to the condition (9) for perfect asymptotic PQSM, Eq. (18) follows from the fact that distillable entanglement cannot increase under LOCC operations.

Pure states. In the final part of the paper we will show that in the task considered here the conditional entropy $S(\rho^{BC}) - S(\rho^{C})$ admits the same interpretation as in standard quantum state merging, i.e., it quantifies the minimal amount of quantum communication needed to achieve perfect merging.

If Bob and Charlie have access to additional entangled states $|D_i\rangle^{B'C'}$ with initial distillable entanglement D_i, perfect PQSM of the state $|\psi\rangle = |\psi\rangle^{ABC}$ can be seen as the following asymptotic transformation:

$$|\psi\rangle^{ABC} \otimes |D_i\rangle^{B'C'} \otimes |0\rangle^{R} \xrightarrow{\Lambda_{\rm PPT}} |\psi\rangle^{ACR} \otimes |D_f\rangle^{B'C'} \otimes |0\rangle^{B}. \tag{19}$$

Here, R is a register in Charlie's possession, and the state $|D_f\rangle^{B'C'}$ has final distillable entanglement D_f. This condition means that by using additional singlets at rate D_i, Bob and Charlie can perfectly merge the state $|\psi\rangle$ in the asymptotic limit via PPT assisted LOCC, and will at the same time gain singlets at rate D_f. The entanglement cost of the process is then given by $D_i - D_f$.

We will now show that the conditional mutual information of the reduced state ρ^{BC} is equal to the minimal entanglement cost of the above process. For this, we note that perfect merging is always possible at cost $D_i - D_f = S(\rho^{BC}) - S(\rho^{C})$, since there exists an LOCC protocol accomplishing this task at this cost [1, 2]. In the following, we will see that PPT assisted LOCC cannot lead to lower cost, i.e.,

$$D_i - D_f \geq S(\rho^{BC}) - S(\rho^{C}) \tag{20}$$

is true for any PPT assisted LOCC protocol achieving perfect merging as in Eq. (19). For proving this, we will introduce the initial state $|\Psi_i\rangle$ and the final state $|\Psi_f\rangle$. They correspond to the total state on the left-hand side and the right-hand side of Eq. (19). Using the fact that for pure states the PPT distillable entanglement $D_{\rm PPT}$ is equal to the von Neumann entropy of the reduced state (see also Eq. (2) and discussion there), it is straightforward to verify the following equality:

$$D_i - D_f = S(\rho^{BC}) - S(\rho^{C}) + D_{\rm PPT}(|\Psi_i\rangle) - D_{\rm PPT}(|\Psi_f\rangle), \tag{21}$$

where the PPT distillable entanglement $D_{\rm PPT}$ is considered with respect to the bipartition $ABB'{:}CC'R$. The desired inequality (20) follows by noting that $D_{\rm PPT}$ cannot increase under PPT assisted LOCC, and thus $D_{\rm PPT}(|\Psi_i\rangle) \geq D_{\rm PPT}(|\Psi_f\rangle)$.

Conclusions. In this paper we introduced and studied the task of PPT quantum state merging (PQSM), where two parties – Bob and Charlie – aim to merge their shares of a tripartite mixed state by using PPT entanglement and classical communication, while preserving the coherence with Alice.

We considered the fidelity of this process, both in the single-copy and the asymptotic scenario, and showed that fully separable states can be perfectly merged asymptotically if and only if they can be perfectly merged on the single-copy level. We used this result to present a family of fully separable states which do not admit perfect asymptotic PQSM. We also identified very general conditions for a state to have vanishing fidelity of PQSM in the asymptotic limit. We showed that these conditions apply to a significant amount of quantum states having nonzero measure in the set of all states, thus proving that a large number of quantum states cannot be merged asymptotically with any nonzero precision. It remains an open question if PQSM exhibits the strong converse property, but we hope that our results can be helpful to prove this statement in the future. For pure states we showed that the conditional entropy admits the same interpretation as in standard quantum state merging: it quantifies the minimal amount of quantum communication needed for perfectly merge the pure state in the asymptotic limit.

We close the discussion by mentioning that the protocol considered here cannot be extended to the scenario where Bob and Charlie have access to arbitrary bound entangled states. In particular, if there exist NPT bound entangled states, the results presented in [17] immediately imply that Bob and Charlie also have access to an unlimited amount of singlets, and thus all states can be perfectly merged. On the other hand, if NPT bound entangled states do not exist, the scenario described here already represents the most general situation.

Acknowledgements. I acknowledge discussion with Andreas Winter. This work was supported by the Alexander von Humboldt-Foundation, John Templeton Foundation, EU IP SIQS, EU Grant QUIC, ERC AdG OSYRIS, and the Spanish Ministry projects FOQUS and DIQIP CHIST-ERA.

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SUPPLEMENTAL MATERIAL

1. Proof of Theorem 1

In the following we will prove that any state $\rho = \rho^{ABC}$ which is PPT with respect to the bipartition $AB : C$ satisfies the inequality

$$\mathcal{F}_{\mathrm{PPT}}(\rho) \geq \mathcal{F}_{\mathrm{PPT}}(\rho^{\otimes n}) \tag{A.22}$$

for any number of copies $n \geq 1$. We will prove this inequality for $n = 2$, and for larger n the proof follows similar lines of reasoning.

For $n = 2$ we will denote the total initial state by

$$\rho \otimes \rho = \rho^{A_1 B_1 C_1} \otimes \rho^{A_2 B_2 C_2}, \tag{A.23}$$

and the final state $\sigma_f = \sigma_f^{A_1 A_2 C_1 C_2 R_1 R_2}$ is then given by

$$\sigma_f = \mathrm{Tr}_{B_1 B_2 \tilde{B} \tilde{C}} \left[\Lambda \left[\rho^{A_1 B_1 C_1} \otimes \rho^{A_2 B_2 C_2} \otimes \mu_{\mathrm{PPT}}^{\tilde{B}\tilde{C}} \otimes \rho^{R_1 R_2} \right] \right], \tag{A.24}$$

where $\mu_{\mathrm{PPT}}^{\tilde{B}\tilde{C}}$ is a PPT state, Λ is an LOCC operation between Bob's total system $B_1 B_2 \tilde{B}$ and Charlie's total system $C_1 C_2 \tilde{C} R_1 R_2$, and $\rho^{R_1 R_2}$ is an arbitrary initial state of Charlie's register.

We will now prove Eq. (A.22) by contradiction, assuming that it is violated for some state ρ which is PPT with respect to $AB : C$. In this case there must exist a PPT state μ_{PPT} and an LOCC protocol Λ such that

$$F(\sigma_f, \sigma_t^{A_1 C_1 R_1} \otimes \sigma_t^{A_2 C_2 R_2}) > \mathcal{F}_{\mathrm{PPT}}(\rho), \tag{A.25}$$

where the final state σ_f was given in Eq. (A.24). The target state $\sigma_t^{A_1 C_1 R_1} \otimes \sigma_t^{A_2 C_2 R_2}$ is the same as $\rho^{A_1 B_1 C_1} \otimes \rho^{A_2 B_2 C_2}$ up to relabeling the parties B_1 and R_1, and B_2 and R_2.

We will now show that Bob and Charlie can "simulate" such a two-copy protocol with just one copy of the state ρ, thus achieving a single-copy fidelity strictly above $\mathcal{F}_{\mathrm{PPT}}$, which will be the desired contradiction. The basic idea of the proof is illustrated in Fig. 2. We assume that Alice, Bob and Charlie start with only one copy of the state $\rho = \rho^{ABC}$, and that the state is PPT between AB and C. Since Bob and Charlie can prepare arbitrary PPT states, they can additionally prepare the state $\rho^{A'B'C'}$, which is equivalent to ρ^{ABC} up to the fact that A' and B' are both in Bob's possession, see Fig. 2.

In the next step, Bob and Charly prepare a PPT state μ_{PPT} and run the same LOCC protocol Λ which was leading to Eq. (A.25). By following this strategy, they will end up with a final state $\sigma_f^{AA'CC'RR'}$ having the property that

$$F(\sigma_f^{AA'CC'RR'}, \sigma_t^{ACR} \otimes \sigma_t^{A'C'R'}) > \mathcal{F}_{\mathrm{PPT}}(\rho). \tag{A.26}$$

Recalling that fidelity does not decrease under discarding subsystems, it follows that

$$F(\sigma_f^{ACR}, \sigma_t^{ACR}) > \mathcal{F}_{\mathrm{PPT}}(\rho), \tag{A.27}$$

which is the desired contradiction.

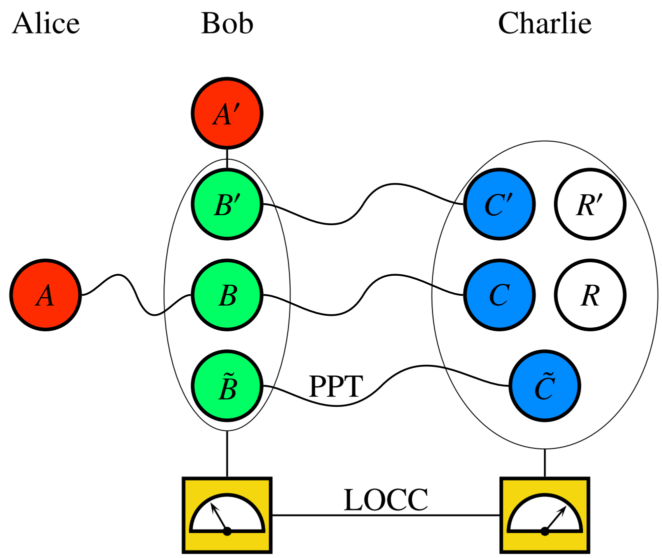

Figure 2. Proof of Eq. (A.22) for $n = 2$. A violation of Eq. (A.22) could be used to build a protocol acting on one copy of the state ρ, and reaching a higher single-copy fidelity than $\mathcal{F}_{\mathrm{PPT}}(\rho)$.

The proof for arbitrary $n \geq 2$ follows by applying the same arguments. Moreover, using the same ideas it is possible to show that the fidelity of LQSM $\mathcal{F}$ satisfies the inequality

$$\mathcal{F}(\rho) \geq \mathcal{F}(\rho^{\otimes n}) \tag{A.28}$$

for any $n \geq 2$ and any state ρ which is separable between AB and C.

2. Fully separable states not admitting perfect asymptotic PQSM

Here we will present a family of fully separable tripartite states $\rho_{\mathrm{sep}}^{ABC}$ that cannot be merged via PPT assisted LOCC even in the asymptotic scenario. The desired family of states is given by

$$\rho_{\mathrm{sep}}^{ABC} = \sum_{i=0}^{14} p_i \left| i \right\rangle \left\langle i \right|^A \otimes \sigma_i^{BC}. \tag{A.29}$$

Here, all states σ_i^{BC} are separable two-qubit states and the particle A has dimension 15 (the reason for this will become clear below). The probabilities p_i are strictly positive for all $0 \leq i \leq 14$.

Note that any general d-dimensional Hilbert space has an associated Bloch vector space of dimension $d^2 - 1$ [19]. In the case considered here, the particles B and C are qubits. Thus, the Bloch vector space associated with the Hilbert space of BC has dimension 15. Moreover, note that there exist 15 separable two-qubit states σ_i^{BC} with the property that all their Bloch vectors are linearly independent. This follows from the fact that the set of separable states has finite size within the set of all states [20].

As we will see in the following, the state in Eq. (A.29) cannot be merged via PPT assisted LOCC whenever the generalized Bloch vectors of the states σ_i^{BC} are linearly independent for all $0 \leq i \leq 14$. Due to Theorem 1 of the main text it is enough to focus on the single-shot scenario, since a fully separable state admits perfect asymptotic PQSM if and only if it admits perfect PQSM in the single-shot scenario.

Using the above result, we will now prove the desired statement by contradiction. Assume that the state $\rho_{\mathrm{sep}}^{ABC}$ with the above properties can be merged with some single-shot PPT assisted LOCC protocol Λ_{PPT} between Bob and Charlie. It then immediately follows that this protocol must merge each of the states σ_i^{BC} individually. Moreover, by convexity, this protocol also merges each convex combination of the form

$$\tau^{BC} = \sum_{i=0}^{14} q_i \sigma_i^{BC}. \tag{A.30}$$

Recall that the set of states of the form (A.30) has finite size within all two-qubit states. By convexity, this implies that the protocol Λ_{PPT} can be used for single-shot merging of *any* state shared by Bob and Charlie. In particular, this means that Λ_{PPT} can merge both states $|00\rangle^{BC}$ and $|{+}0\rangle^{BC}$. The existence of such a protocol would thus imply that the states $|0\rangle$ and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ can be perfectly teleported with the aid of PPT states on the single-copy level. This is however impossible [21], which is the desired contradiction. This completes the proof that the aforementioned family of states does not admit perfect asymptotic PQSM.

3. States with vanishing asymptotic fidelity

Here we will show that all states satisfying the inequality

$$D_{\mathrm{LOCC}}^{A:BC}(\rho) > E_n^{AB:C}(\rho) = 0 \tag{A.31}$$

have zero fidelity in the asymptotic limit:

$$\mathcal{F}_{\mathrm{PPT}}^{\infty}(\rho) = 0. \tag{A.32}$$

For this we note that for all such states the final state $\sigma_f = \sigma_f^{ACR}$ is PPT with respect to the bipartition $A{:}CR$, and thus is nondistillable with respect to this bipartition. This means that for any number of copies n the fidelity of PQSM is bounded above as follows:

$$\mathcal{F}_{\mathrm{PPT}}(\rho^{\otimes n}) = \sup_{\Lambda_{\mathrm{PPT}}} F(\sigma_t^{\otimes n}, \sigma_f) \leq \sup_{\tau \in \overline{\mathcal{D}}} F(\sigma_t^{\otimes n}, \tau), \tag{A.33}$$

where the final state σ_f shared by Alice and Charlie is given as $\sigma_f = \mathrm{Tr}_B[\Lambda_{\mathrm{PPT}}[\rho^{\otimes n} \otimes \rho^R]]$, and the supremum in the last expression is taken over all states τ which are not distillable between Alice and Charlie.

In the next step, we introduce the geometric distillability

$$D_g(\nu) = 1 - \sup_{\tau \in \overline{\mathcal{D}}} F(\nu, \tau), \tag{A.34}$$

and note that the target state $\sigma_t = \sigma_t^{ACR}$ in Eq. (A.33) is distillable between Alice's system A and Charlie's system CR. For proving Eq. (A.32) it is thus enough to show that for any distillable state ν the geometric distillability approaches one in the asymptotic limit:

$$\lim_{n\to\infty} D_g(\nu^{\otimes n}) = 1. \tag{A.35}$$

Surprisingly, this is indeed the case for any distillable state ν, and the proof will be given in the following.

4. Asymptotic geometric distillability

In the following we consider the geometric distillability defined as

$$D_g(\rho) = 1 - \sup_{\sigma \in \overline{\mathcal{D}}} F(\rho, \sigma), \tag{A.36}$$

where $F(\rho, \sigma) = \mathrm{Tr}(\sqrt{\rho}\sigma\sqrt{\rho})^{1/2}$ is the fidelity, and the supremum is taken over the set of nondistillable states $\overline{\mathcal{D}}$. We will also consider the closely related quantity

$$D_t(\rho) = \inf_{\sigma \in \overline{\mathcal{D}}} T(\rho, \sigma), \tag{A.37}$$

where $T(\rho, \sigma) = ||\rho - \sigma||/2$ is the trace distance with the trace norm $||M|| = \mathrm{Tr}\sqrt{M^\dagger M}$. The trace distance and fidelity are related as

$$1 - F(\rho, \sigma) \leq T(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}. \tag{A.38}$$

As we will show in the following, both quantities D_g and D_t are discrete in the asymptotic limit: asymptotically they attain only the values 0 (if ρ is nondistillable) and 1 (if ρ is distillable). For nondistillable states ρ it is clear that D_g and D_t are both zero, and thus also zero asymptotically. We will now prove the following equality for any distillable state ρ:

$$\lim_{n\to\infty} D_g(\rho^{\otimes n}) = \lim_{n\to\infty} D_t(\rho^{\otimes n}) = 1. \tag{A.39}$$

Note that due to Eq. (A.38) it is enough to prove only one of the equalities. In the following, we will prove the equality for D_t.

In the first step, we note that Eq. (A.39) is true for the maximally entangled state $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. This can be seen by noting that the fidelity between $|\phi^+\rangle^{\otimes n}$ and any nondistillable state $\sigma \in \overline{\mathcal{D}}$ is bounded above as follows [22, 23]:

$$F(|\phi^+\rangle\langle\phi^+|^{\otimes n}, \sigma) \leq \frac{1}{2^{n/2}}. \tag{A.40}$$

In the next step, note that for a distillable state ρ there exist a sequence of LOCC protocols Λ_n acting on n copies of the state ρ such that

$$\lim_{n\to\infty} T\left(\Lambda_n[\rho^{\otimes n}], |\phi^+\rangle\langle\phi^+|^{\otimes\lfloor nE_d\rfloor}\right) = 0, \tag{A.41}$$

where E_d is the distillable entanglement or ρ and $\lfloor x \rfloor$ is the largest integer below x. Moreover, without loss of generality,

we assume that $\Lambda_n[\rho^{\otimes n}]$ and $|\phi^+\rangle^{\otimes\lfloor nE_d\rfloor}$ have the same dimension.

By applying the triangle inequality with some nondistillable state σ we further obtain:

$$T\left(|\phi^+\rangle\langle\phi^+|^{\otimes\lfloor nE_d\rfloor},\sigma\right) \leq T\left(\Lambda_n[\rho^{\otimes n}], |\phi^+\rangle\langle\phi^+|^{\otimes\lfloor nE_d\rfloor}\right) + T\left(\Lambda_n[\rho^{\otimes n}],\sigma\right). \tag{A.42}$$

Minimizing both sides of this inequality over all nondistillable states σ, it follows that:

$$D_t\left(|\phi^+\rangle^{\otimes\lfloor nE_d\rfloor}\right) \leq T\left(\Lambda_n[\rho^{\otimes n}], |\phi^+\rangle\langle\phi^+|^{\otimes\lfloor nE_d\rfloor}\right) + D_t\left(\Lambda_n[\rho^{\otimes n}]\right). \tag{A.43}$$

In the final step, we take the limit $n \to \infty$ and use Eq. (A.41), arriving at the following result:

$$\lim_{n\to\infty} D_t\left(|\phi^+\rangle^{\otimes n}\right) \leq \lim_{n\to\infty} D_t\left(\Lambda_n[\rho^{\otimes n}]\right). \tag{A.44}$$

Recalling the fact that Eq. (A.39) is true for the maximally entangled state $|\phi^+\rangle$, this inequality implies

$$\lim_{n\to\infty} D_t\left(\Lambda_n[\rho^{\otimes n}]\right) \geq 1. \tag{A.45}$$

The proof of Eq. (A.39) for all distillable states is complete by noting that D_t cannot increase under LOCC, i.e., $D_t(\rho^{\otimes n}) \geq D_t(\Lambda_n[\rho^{\otimes n}])$.

5. States with vanishing asymptotic fidelity have nonzero measure

We will now show that the set of states with vanishing asymptotic fidelity has nonzero measure in the set of all states. For this we will present a family of three-qubit states $\rho = \rho^{ABC}$ which are separable between AB and C, do not touch the boundary of separable states, and are distillable between A and BC. This assures that small perturbations of this state do not change its basic properties, i.e., the perturbed states are also separable between AB and C, distillable between A and BC, and thus have vanishing asymptotic fidelity $\mathcal{F}^{\infty}_{\mathrm{PPT}}(\rho) = 0$.

The following three-qubit state has the aforementioned properties:

$$\rho = (1-p)\,|\phi^+\rangle\langle\phi^+| \otimes |0\rangle\langle 0| + p\frac{\mathbb{1}}{8} \tag{A.46}$$

with $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. The parameter p can be chosen from the range $0 < p < p_{\max}$, and $p_{\max} > 0$ is chosen such that the state ρ is distillable between A and BC for all $p < p_{\max}$.

In order to see that the state obtained in this way is not on the boundary of separable states (with respect to the bipartition AB:C), we consider a small perturbation of the form

$$\rho' = \varepsilon\sigma + (1-\varepsilon)\rho \tag{A.47}$$

with an arbitrary three-qubit state σ. The proof is complete by noting that for any σ there exists some maximal parameter $\varepsilon_{\max}(\sigma) > 0$ such that ρ' is separable for all $0 \leq \varepsilon \leq \varepsilon_{\max}(\sigma)$. This follows directly from the existence of a separable ball around the maximally mixed state [20].